Lévy Computability of Probability Distribution Functions

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0. Introduction

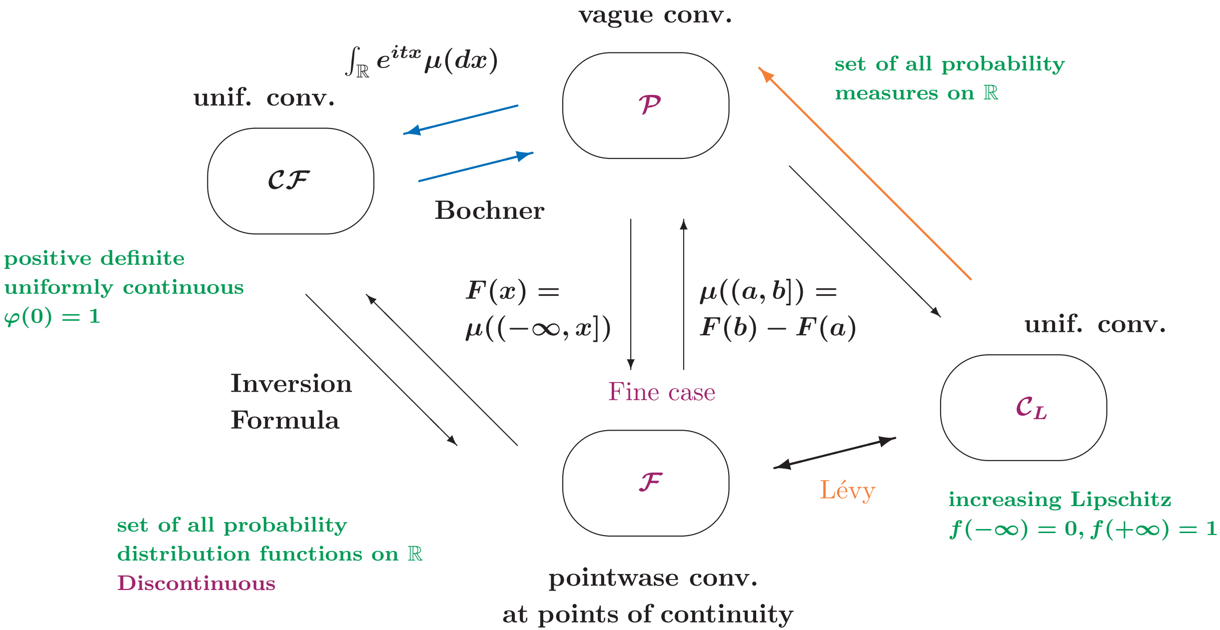


Figure 1: **The spaces $\mathcal{P}, \mathcal{F}, \mathcal{C}_L$**

Notations: $\mu \leftrightarrow F \leftrightarrow \mathcal{L}F = f$ (unless oterwise stated)

Computabilities on $\mathcal{F}$

We had treated Fine computability and effective Fine convergence

We seek: computability and effective convergence on $\mathcal{F}$
relations to Fine computabilities on $\mathcal{F}$
relations to computability and effective convergence on $\mathcal{P}$

Translated Dirac measure $\delta_{\frac{1}{3}}$ is computable, but the corresponding probability distribution functions is not Fine computable.

We have an example of probability distribution function F such that $F(0)$ is not computable, but the corresponding μ is computable
(Example 4.2)

We propose Lévy computability and effective Lévy convergence on $\mathcal{F}$.

♡

Convergence on $\mathcal{P}$

motivation

$$\boxed{\textbf{convergence} \leftrightarrow \textbf{determining class} \leftrightarrow \textbf{computability}}$$

a class $\mathcal{A}$ of functions or of sets
is a determining class if $\mu(\eta) = \nu(\eta)$ for $\forall \eta \in \mathcal{A}$ implies $\mu = \nu$

1. Classical Theory about probability measures on $\mathbb{R}$ (Summary)

Probability Distribution Functions are characterized by

(F-i) monotonically increasing

(F-ii) right-continuous

(F-iii) $F(\infty) = 1$, $F(-\infty) = 0$

Notations:

$\mathcal{F}$: the set of all probability distribution functions

$\mathcal{C}_\kappa$: the set of all continuous functions with compact support

$\mathcal{C}_b$: the set of all bounded continuous functions

$\mu(f) = \int_{\mathbb{R}} f(x)\mu(dx)$

$\{g_\ell\}$: a dense sequence of $\mathcal{C}_\kappa$

$\{g_\ell\}$-metric : $d_{\{g_\ell\}}(\mu, \nu) = \sum_{\ell=1}^{\infty} 2^{-\ell}(|\mu(g_\ell) - \nu(g_\ell)| \wedge 1)$

The following converences are equivalent ([1], [6]).

(i)	$\mu_m(f) \to \mu(f)$	$\forall f \in \mathcal{C}_\kappa$	(vague)
(ii)	$\mu_m(f) \to \mu(f)$	$\forall f \in \mathcal{C}_b$	(weak)
(iii)	$\liminf_m \mu_m(G) \geq \mu(G)$	$\forall G$: open	domain theory S2007[23]
(iv)	$d_{\{g_\ell\}}(\mu_m, \mu) \to 0$	$\{g_\ell\}$: dense in $\mathcal{C}_\kappa$	W1999[27][0,1]

$[0, 1]$

(iii′)	$\liminf_m \mu_m(I) \geq \mu(I)$	$\forall I$: open interval	W1999[27]
		Intervals	SS2005[22]

$\mathbb{R}$

(v)	$\varphi_m(t) \to \varphi(t)$	for any t	
(vi)	$F_m(x) \to F(x)$	$\forall x$ a point of continuity of F $\Leftrightarrow \mu(\{x\}) = 0, \quad \{(-\infty, x]\}$	
(vii)	$d_L(F_m, F) \to 0$		Lévy

(other metrices: cf. [2])

♡ ◇

2. Lévy height and Lévy metric

Lévy [11], Ito [6]

Notations:

$$\mathcal{G}_F = \{(x,y) \mid F_-(x) \le y \le F(x)),\ x \in \mathbb{R}\} \qquad (F_-(x) = \lim_{y\uparrow x} F(y)\ )$$

$\forall t \in \mathbb{R}$, $(x(t), y(t))$ = the unique crossing point of $X + Y = t$ with $\mathcal{G}_F$

Definition 2.1 Lévy height: $\mathcal{L}F(t) = y(t)$

Definition 2.2 Lévy metric (distance): $d_L(F,G) = \sup_{t\in\mathbb{R}} \left|\mathcal{L}F(t) - \mathcal{L}G(t)\right|$

Definition 2.3 Lévy convergence: $d_L(F_n, F) \to 0$

Remark 2.4 $d_L(F,G)$ is equal to the Lévy(-Prokhorov) metric, that is,

$$d_L(F,G) = \inf\{\epsilon > 0 \mid F(x-\epsilon) - \epsilon \le G(x) \le F(x+\epsilon) + \epsilon,\ \textit{for all } x\}$$

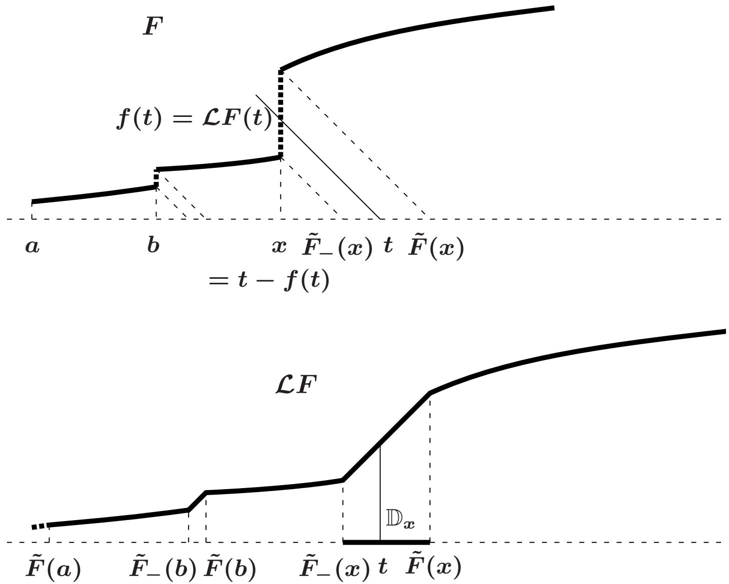


Figure 2: F and $\mathcal{L}F$

Put $\mathcal{L}F(t) = f(t)$.

$\mathbb{D}_x = \{t \mid x = t - f(t)\}$

$\sup\{f(t) \mid t \in \mathbb{D}_x\} - \inf\{f(t) \mid t \in \mathbb{D}_x\} =$ **the jump of** F **at** x

♡

Properties of Lévy height

Let $\mathcal{C}_L$ be the set of functions which satisfy the following:

(Li) $0 \le f(t) - f(s) \le t - s$ if $s \le t$.

(Lii) $\lim_{t\to-\infty} f(t) = 0$.

(Liii) $\lim_{t\to\infty} f(t) = 1$.

Proposition 2.5 $\mathcal{L}$: $\mathcal{F} \to \mathcal{C}_L$ *one-to-one and onto*

$\mathcal{C}_L$: closed convex subset of $\mathcal{C}_b$ w.r.t. the sup-norm (distance) d_∞

Hence, $(\mathcal{C}_L, d_\infty)$ is complete

Properties of $\tilde{F}$, $\tilde{F}_-$, $\mathcal{L}F$ $\qquad$ $\mathbb{D}_x = [\tilde{F}_-(x), \tilde{F}(x)]$

- $\tilde{F}(x) = x + F(x)$ is strictly increasing and continuous
  $\mathcal{L}F(t) = F(\tilde{F}^{-1}(t))$
- $[\tilde{F}_-(x), \tilde{F}(x)) \cap Range(F) = \phi$
- $\tilde{F}^{-1}(t)$ is computable if F is computable
- $F(x) = f(\sup \mathbb{D}_x)$
- $\hat{f}(t) = t - f(t)$ is a nondecreasing continuous function from $\mathbb{R}$ onto $\mathbb{R}$

Example 2.6 Dirac measures δ_a, D_a: prof. dist. function

- $D_a(x) = 0$ if $x < a$ and $= 1$ if $x \geq a$

- $\mathcal{L}D_a(t) = \begin{cases} 0 & \text{if } \; t \leq a \\ t - a & \text{if } \; a \leq t \leq 1 + a \\ 1 & \text{if } \; t \geq 1 + a \end{cases}$

- $d_L(D_a, D_b) = d_\infty(\mathcal{L}D_a, \mathcal{L}D_b) = |a - b| \wedge 1$

3. Computability on $\mathcal{P}$

We employ computable notions on $\mathbb{R}$ by Pour-El and Richards.

Definition 3.1 $\{\mu_m\}$ is **computable**

$\leftrightharpoons$ $\{\mu_m(f_n)\}$ is computable for any computable sequence f_n
with compact support
($L(n)$ s.t. $f_n(x) = 0$ if $|x| \geq L(n)$ for some recursive $L(n)$)

Definition 3.2 $\{\mu_m\}$ **converges effectively** to μ

$\leftrightharpoons$ $\{\mu_m(f_n)\}$ converges effectively to $\{\mu(f_n)\}$
for any computable $\{f_n\}$ with recursive compact support

Notation: $\{\mu_m\} \xrightarrow{e} \mu$

weakly: computable sequence f_n with compact support

$\longrightarrow$ effectively bounded computable $\{\mu(f_n)\}$

$\exists M(n)$: recursive such that $|f_n(x)| \leq M(n)$

Notation: $\{\mu_m\} \xrightarrow{ew} \mu$

Proposition 3.3

(1) $\{\mu_m\}$ *is computable* $\Leftrightarrow$ $\{\mu_m\}$ *is weakly computable*

(2) *Assume* $\{\mu_m\}$ *and* μ *are computable. Then,*

$$\{\mu_m\} \xrightarrow{e} \mu \Leftrightarrow \{\mu_m\} \xrightarrow{ew} \mu$$

4. Lévy computabilities

Definition 4.1 $\{F_n\}$ is said to be **Lévy computable** $\leftrightharpoons$ $\{\mathcal{L}F_n\}$ is computable.

Intuitively,
Lévy computability means that we can draw the graph $\mathcal{G}_F$ effectively.

That is, $(\hat{f}(t), f(t))$ is a one parametric representation of $\mathcal{G}_F$, in the sense of Skotokhod [24]. $(\hat{f}(t) = t - f(t))$

(Section 6, P. 21)

Next Example shows that there exists a probability measure ν:

ν **is computable**

$G(0)$ **is not a computable real**

Example 4.2 (Example 3.4 in [16])

α: one-to-one recursive with non-recursive range.

$\boldsymbol{d = \sum_{i=1}^{\infty} 2^{-\alpha(i)} < 1, \quad d_n = \sum_{i=1}^{n} 2^{-\alpha(i)} \quad (d_0 = 0)}$

$$\boldsymbol{\nu = (1-d)\delta_0 + \sum_{i=1}^{\infty} 2^{-\alpha(i)}\delta_{2^{-(i-1)}}, \quad \nu_n = (1-d_n)\delta_0 + \sum_{i=1}^{n} 2^{-\alpha(i)}\delta_{2^{-(i-1)}}}$$

G and $\{G_n\}$: the corresponding probability distribution functions

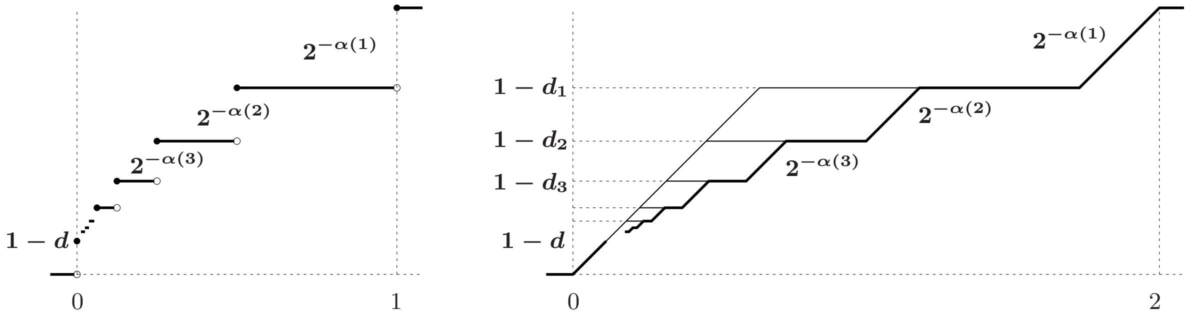


Figure 3: **Graph of G and $\mathcal{L}G$**

The followings holds:

- $\{G_n\}$: **monotonically decreasing w.r.t.** n, **and converges to** G
- $\{\mathcal{L}G_n\}$: **monotonically decreasing w.r.t.** n, **and converges to** $\mathcal{L}G$
- $\mathcal{L}G(t) = \mathcal{L}G_n(t)$ **if** $t \leq 1 - d$ **or** $t \geq 1 - d_{n+1} + 2^{-(n+1)}$
  $\neq$ **on** $(1 - d, 1 - d_{n+1} + 2^{-(n+1)})$
- $\{\mathcal{L}G_n\}$ **is computable**

We can prove that $\{\mathcal{L}G_n\}$ **converges to** $\mathcal{L}G$ **effectively uniformly.**

This implies that $\mathcal{L}G$ **is computable and** G **is Lévy computable.**

G **is not continuous, Fine continuous,**
not Fine computable, Lévy computable.

♡

Theorem 4.3 *Assume that $\{F_m\}$ is continuous. Then*

$$\{F_m\} \textit{ is Lévy computable} \Leftrightarrow \{F_m\} \textit{ is computable.}$$

Proposition 4.4 $\mu(g) = \int_{\mathbb{R}} g(x)dF(x) = \int_{\mathbb{R}} g(\hat{f}(t))df(t)$ *for all* $g \in \mathcal{C}_\kappa$

Theorem 4.5 *If $\{F_m\}$ is Lévy computable, then $\{\mu_m\}$ is computable.*

Lemma 4.6 *D_a: the probability distribution function of δ_a*

$\mathcal{L}F$ is computable, p is positive omputable, a is computable, then

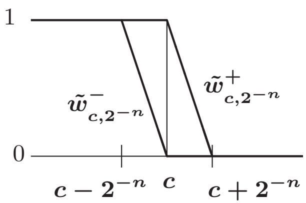
(1) $\mathcal{L}(pD_a)$ *is computable.*

(2) $\mathcal{L}(pD_a + F)$ *is computable.*

(3) $\mathcal{L}F(t) \leq \mathcal{L}(pD_a + F)(t) \leq p + \mathcal{L}F(t)$.

$\mathcal{P} \to \mathcal{C}_L$ $$\mu(g) = \int_{\mathbb{R}} g(x)dF(x) = \int_{\mathbb{R}} g(\hat{f}(t))df(t)$$

$\mathcal{P} \to \mathcal{F}$



$$\mu(\tilde{w}^{-}_{c,n}) \leq F(c) = \mu(\chi_{(-\infty.c]}) \leq \mu(\tilde{w}^{+}_{c,n})$$

Figure 4: $\tilde{w}^{+}_{c,n}(x)$ and $\tilde{w}^{-}_{c,n}(x)$

If μ is computable and c is computable,
then $\mu(\tilde{w}^{-}_{c,n})$ and $\mu(\tilde{w}^{+}_{c,n})$ are computable.

$F(c)$ is right (lower) computable
but we cannot derive the computability of $F(c)$ (+ continuity)

5. Lévy Computability and Fine Computability for Probability Distribution Functions

Fine topology is generated by $\{I(k,i) = [\frac{i}{2^k}, \frac{i+1}{2^k}) \mid k \in \mathbb{N}, i \in \mathbb{Z}\}$
$J(x,k)$ is the unique $I(k,i)$ which contains x.
Fine computability is defined with respect to this Fine topology.
$\{e_i\}$ is an effective enumaration of dyadic rationals.

Definition 5.1 A sequence of functions $\{f_n\}$ is said to be Fine computable if it satisfies

(i) (*Sequential Fine computability*)
$\{f_n(x_m)\}$ is computable for any Fine computable $\{x_m\}$

(ii) (*Effective Fine Continuity*)
There exists a recursive function $\alpha(n,k,i)$ such that

(ii-a) $x \in J(e_i, \alpha(n,k,i)) \Rightarrow |f_n(x) - f_n(e_i)| < 2^{-k}$

(ii-b) $\bigcup_{i=1}^{\infty} J(e_i, \alpha(n,k,i)) = [0,1)$ for each n, k

uniformly Fine computable if $\alpha(n,k,i)$ does not depend on i

Theorem 5.2 *$\{F_m\}$ is Fine computable $\Rightarrow$ $\{F_m\}$ is Lévy computable*

First, we prove the following special case.

Proposition 5.3 *$\{F_m\}$ is uniformly Fine computable*
$\Rightarrow$ $\{F_m\}$ is Lévy computable.

<u>Outline of the proof of Proposition 5.3</u>

Lemma 5.4 *If $\{\mu_m\}$ is computable,*
then there exists a recursive function $L(m,k)$ such that
$\mu_m(w_n) > 1 - 2^{-k}$, equivalently $\mu(w_n^c) < 2^{-k}$, for all $n \geq L(m,k)$.

(For a single F) $\alpha(k)$: mopdulus of effective uniform Fine continuity
note: $F_{-}(x)$ is computable if x is Fine computable

♡

Let $\{t_n\}$ be computable

$\forall k$, the set of fite open intervals

$$(-\infty, -L(k) + 2^{-k}),\ (L(k) - 2^{-k}, \infty),$$

$$(\tilde{F}(-L(k) + i2^{-\alpha(k)} - 2^{-k}, \tilde{F}_{-}(-L(k) + (i+1)2^{-\alpha(k)} + 2 \cdot 2^{-k})$$

$$(0 \leq i \leq 2 \cdot L(k)2^{\alpha(k)})$$

$$(\tilde{F}_{-}(-L(k) + i2^{-\alpha(k)}) - 2^{-k}, \tilde{F}(-L(k) + i2^{-\alpha(k)}) + 2^{-k})$$

$$(\tilde{F}(-L(k) + i2^{-\alpha(k)}) - \tilde{F}_{-}(-L(k) + i2^{-\alpha(k)}) > 2^{-k})$$

is an open covering of $\mathbb{R}$

We can define effectively a sequence $\{r_{n,k}\}$ such that

$|f(t_n) - r_{n,k}| < 2^{-k}$ □

Outline of the proof of Theorem 5.2

Proposition 5.5 ([15]) ***Let $\{f_m\}$ be a Fine computable sequences of functions. Define***

$$\varphi_{m,n}(x) = \sum_{j=0}^{2^n-1} f_m(j2^{-n})\chi_{I(n,j)}(x).$$

Then, $\{\varphi_{m,n}\}$ Fine converges effectively to $\{f_m\}$.

We can prove that $\{\mathcal{L}\varphi_n\}$ converges effectively uniformly to $\mathcal{L}F$.

6. Effective Lévy Convergence

Definition 6.1 $\{F_m\}$ is said to Lévy converge effectively to F
$\overset{\text{def}}{\iff}$ $d_L(F_m, F)$ converges effectively to zero $\qquad \{F_m\} \xrightarrow{eL} F$

Special case of Skorokhod M_1-convergence for GADLAC

A pair of functions $(\lambda(t), \xi(t))$ is said to be a parametric representation of the graph of $\mathcal{G}_F = \{(x, z) \mid F_-(x) \le z \le F(x),\ x \in \mathbb{R}\}$ if $\mathcal{G}_F = \{(\lambda(t), \xi(t) \mid t \in \mathbb{R}\}$, $\xi(t)$ is continuous and $\lambda(t)$ is continuous and monotonically increasing. $\qquad ((\hat{f}(t), f(t)))$

$\{F_m\}$ M_1 converges to F: if there exist a parametric representation $(\lambda(t), \xi(t))$ of $\mathcal{G}_F$ and a sequence of parametric representations $(\lambda_m(t), \xi_m(t))$ of $\{\mathcal{G}_{F_m}\}$ respectively, such that

$$\lim_{n\to\infty} \left\{ \sup_t |\xi_m(t) - \xi(t)| + \sup_t |\lambda_m(t) - \lambda(t)| \right\} = 0$$

Theorem 6.2 *Let* $\{\mu_m\}$, $\mu \in \mathcal{P}$. *Then* $\{F_m\} \xrightarrow{eL} F \Rightarrow \{\mu_m\} \xrightarrow{e} \mu$

use Proposition 4.4 $\quad \mu(g) = \int_{\mathbb{R}} g(x)dF(x) = \int_{\mathbb{R}} g(\hat{f}(t))df(t)$

Theorem 6.3 *Let* $\{\mu_m\}$, $\mu \in \mathcal{P}$ *and* $\{F_m\}$ *be Lèvy computable. Then* $\{F_m\} \xrightarrow{eL} F \Rightarrow \mu$ *is computable*

Definition 6.4 (Effective d-irrationally pointwise Fine convergence, [15])
Let $\{F_m\}$, F be sequentially Fine computable.
$\{F_m\}$ converges effectively d-irrationally pointwise Fine to F
$\leftrightharpoons$ $\{F_m(x_n)\}$ converges effectively to $\{F(x_n)\}$
for any Fine computable d-irrational sequence $\{x_n\}$.

Theorem 6.5 ([16]) *Let $\{F_m\}$, F be sequentially Fine computable. Assume further that F is effectively Fine continuous. Then, effective convergence of $\{\mu_m\}$ to μ is equivalent to effective d-irrationally pointwise Fine convergence of $\{F_m\}$ to F.*

Theorem 6.6 *Let $\{F_m\}$, F be Fine computable. Then,*
$\{F_m\} \xrightarrow{eL} F \Leftrightarrow \{F_m\}$ *effectively d-irrationally pointwise converges to* F

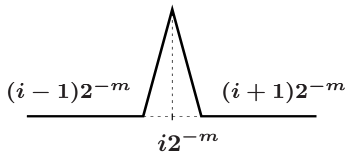
7. $\{g_\ell\}$-metric on $\mathcal{P}$

Proposition 7.1 *Let $\{g_\ell\}$ be a computable sequence in $\mathcal{C}_\kappa$ and $\{\mu_m\}, \{\nu_m\}$ be computable sequences of probability measures. Then, $\{d_{\{g_\ell\}}(\mu_m, \nu_n)\}$ is computable (double) sequence of reals.*

Proposition 7.2 *Let $\{g_\ell\}$ be an effective separating set. Then, effective convergence is equivalent to effective $\{g_n\}$-convergence.*

Theorem 7.3 *Let $\mathcal{S}$ be the set of all computable sequence of probability measures. Then, $(\mathcal{P}, d_{\{g_\ell\}}, \mathcal{S})$ is a metric space with a computability structure in the sense of* (cf. Definition [13])*.*

Example 7.4 $\{g_\ell\}$: an example of an effective separating set in $\mathcal{C}_\kappa$.



$(t_{m,i} = 2^{-m} u_{m,i}$ Weihrauch$)$

Figure 5: **Graph of $u_{m,i}$**

an effective enumeration of the set of all finite linear combinations with rational coefficients of $\{u_{m,i}\}_{m\in\mathbb{N},i\in\mathbb{Z}}$.

Example 7.5 Example of an effective separating set in $(\mathcal{P}, d_{\{g_\ell\}})$

an effective enumeration of the set of all finite convex combinations with rational coefficients of $\{\delta_{i2^{-m}}\}$

Proposition 7.6 *Define $\mu_m = \sum_{i=0}^{2m\,2^{2m}} \mu(u_{m,-m+i2^{-m}})\delta_{-m+i2^{-m}}$. Then, $\{\mu_m\}$ converges to μ.*
Moreover, $\{\mu_m\}$ is computable and converges effectively to μ, if μ is computable.

Definition 7.7 (Effective compactness, [13])
$(X, d, \mathcal{S})$ is said to be effectively totally bounded if there exist an effective separating set $\{e_n\}$ and a recursive function α such that

$$X = \bigcup_{n=1}^{\alpha(p)} \mathbb{B}_X(e_n, 2^{-p}) \quad \text{for all} \quad p.$$

If $(X, d, \mathcal{S})$ is effectively totally bounded and effectively complete, then we say that $(X, d, \mathcal{S})$ is effectively compact.

Proposition 7.8 *$(\mathcal{P}([0,1]), d_{\{g_\ell\}})$ is effectively compact.*

Weihrauch [27] had used $t_{n,m} = 2^{-n} u_{n,m}$ and defined the representation δ''_m. The metric space $(\mathcal{P}([0,1]), \rho)$ is equivalent to $(\mathcal{P}([0,1]), d_{v,\{g_p\}})$. He also defined representations δ_m and δ'_m and proved these representations are equivalent. (Theorem 4.2 in [27]).
By Kamo [8], $d_{\{g_p\}}$-computability is equivalent to δ''_m-computability.

8. Comments on Proofs

Proposition ***If $\{\mu_m\}$ is computable and converges effectively to μ, then μ is computable.***

The following Lemmas and Proposition are used many times.

Lemma 8.1 (Monotone Lemma, [19])
Let $\{x_{n,k}\}$ be a computable sequence of reals which converges monotonically to $\{x_n\}$ as k tends to infinity for each n.
Then, $\{x_n\}$ is computable if and only if the convergence is effective.

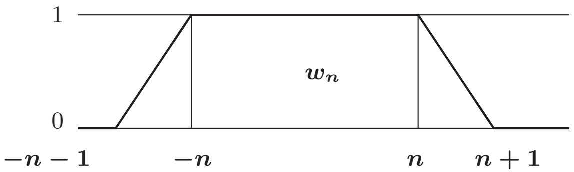


Figure 6: **Graph of w_n**

The following Proposition is fundamental.

Proposition 8.2 (Effective tightness of an effectively convergent sequence, Effectivization of Lemma 15.4 in [25])
If a computable $\{\mu_m\}$ effectively converges to μ,
then there exists a recursive function $\alpha(k)$ such that $\mu_m(w^c_{\alpha(k)}) < 2^{-k}$ for all m.
It also hold that $\mu_m([-\alpha(k)-1, \alpha(k)+1]^C) < 2^{-k}$ for all m.

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9. Characteristic Functions (CCA2012)

Theorem 9.1 *If $\{\mu_m\}$ is computable, then $\{\varphi_m\}$ is uniformly computable.*

Theorem 9.2 (Effective Glivenko, cf, Theorem 2.6.4 in Ito [6])
Let $\{\varphi_m\}$ and φ be computable. Then, $\{\mu_m\}$ converges effectively to μ if $\{\varphi_m\}$ converges effectively to φ .

Theorem 9.3 (Effectivization of Theorem 2.6.3 in Ito [6]) *Let $\{\mu_m\}$ and μ be computable. Then, $\{\varphi_m\}$ converges effectively (compact-uniformly) to φ if $\{\mu_m\}$ converges effectively to μ.*

Theorem 9.4 *$\{\mu_m\}$ is computable if $\{\varphi_m\}$ is computable.*

Theorem 9.4 is the converse to Theorem 9.1. So, we obtain:

Theorem *$\{\mu_m\}$ is computable if and only if $\{\varphi_m\}$ is computable.*

Theorem 9.5 (Effective Bochner's theorem) *In order for $\varphi(t)$ to be a characteristic function of a computable probability measure, it is necessary and sufficient that the following three conditions holds.*
(i) *φ is positive definite.* (ii) *φ is computable.* (iii) $\varphi(0) = 1$.

10. De Moivre-Laplace Central Limit Theorem (CCA2012)

The Central Limit Theorem is one of important theorems in probability theory and in statitics.

Let $(\Omega, \mathcal{B}, \mathbb{P}, \{X_m\})$ be a realization of Coin Tossing (Bernoulli Trials) with success probability p.

Theorem 10.1 (Effective de Moivre-Laplace)
If p is a computable real number, then the sequence of probability measures of random variables

$$Y_m = \frac{X_1 + \cdots X_m - mp}{\sqrt{mp(1-p)}} = \sum_{\ell=1}^{m} \frac{X_\ell - p}{\sqrt{mpq}}$$

converges effectively to the standard Gaussian probability measure.

$$\psi_m(t) = \mathbb{E}(e^{itY_m}) = \prod_{\ell=1}^{m} \mathbb{E}(e^{it\,\frac{X_\ell - p}{\sqrt{mpq}}}) = (pe^{\frac{it\sqrt{q}}{\sqrt{mp}}} + qe^{-\frac{it\sqrt{p}}{\sqrt{mq}}})^m$$

By Theorems 10.1 and Theorem 9.2, the following hold:

$$\mathbb{E}(f(Y_m)) \to \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) x 2^{-\frac{t^2}{2}} dt$$

effectively if f is bounded computable.

11. Graphs

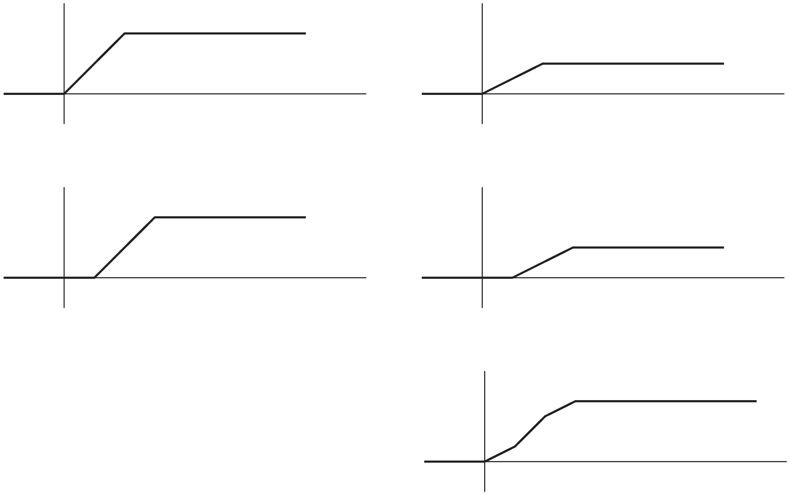


Figure 7: $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{\frac{1}{2}}$

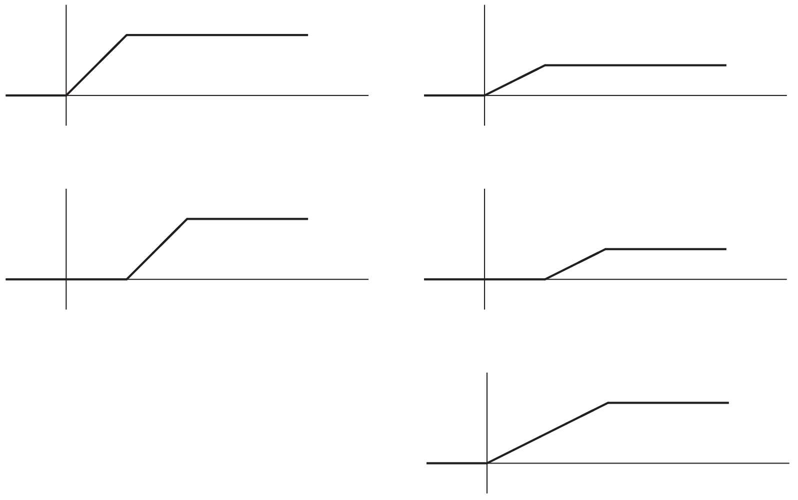


Figure 8: $\frac{1}{2}\boldsymbol{\delta_0} + \frac{1}{2}\boldsymbol{\delta_1}$

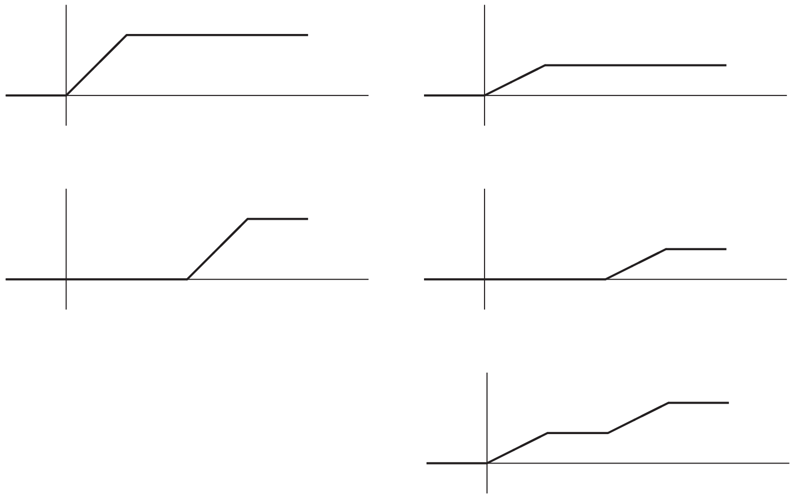


Figure 9: $\frac{1}{2}\boldsymbol{\delta_0} + \frac{1}{2}\boldsymbol{\delta_2}$

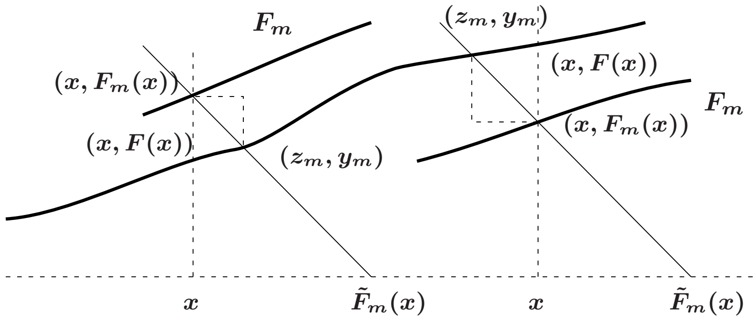


Figure 10: Distance between $F(x)$ and $F_m(x)$

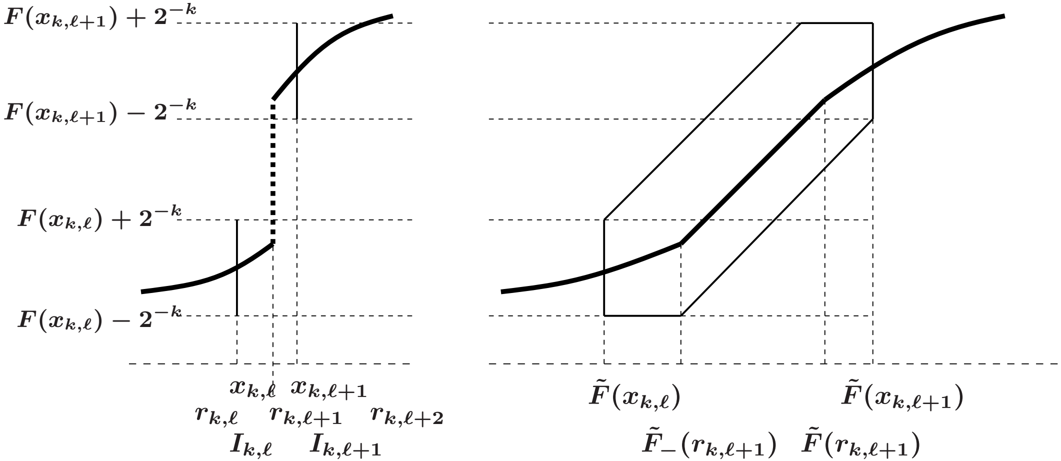


Figure 11: Lévy convergence

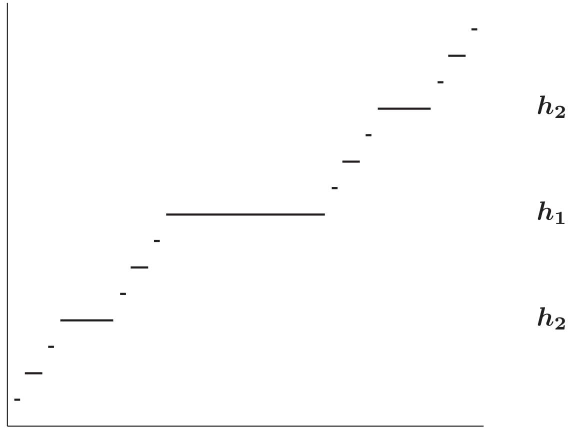


Figure 12: Graph of singular $\boldsymbol{G}$

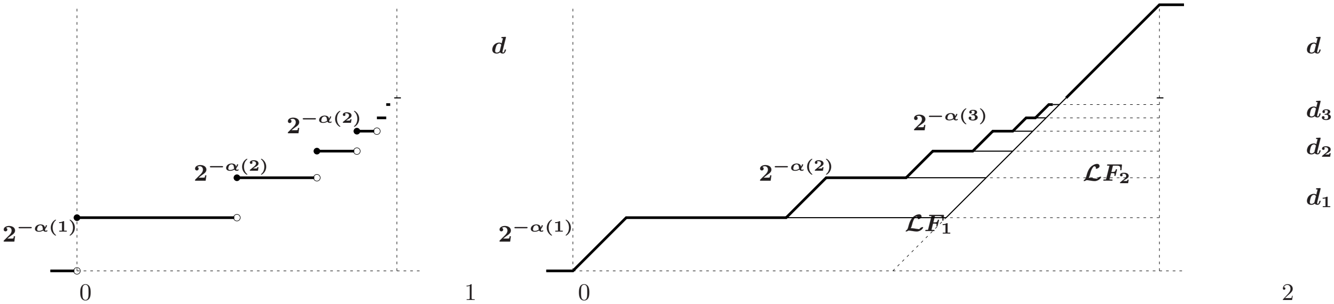


Figure 13: **Graph of F and $\mathcal{L}F$**

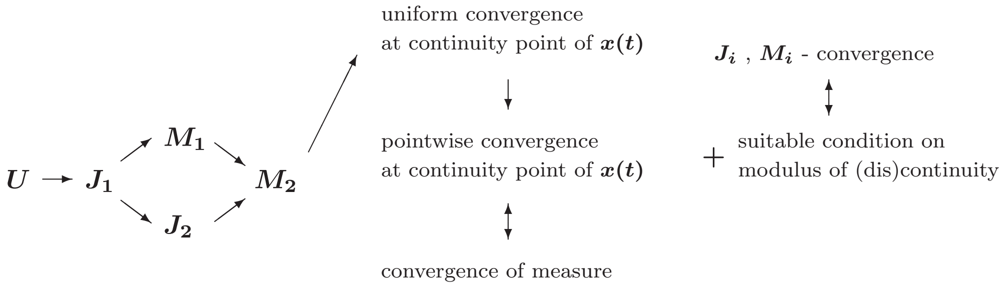
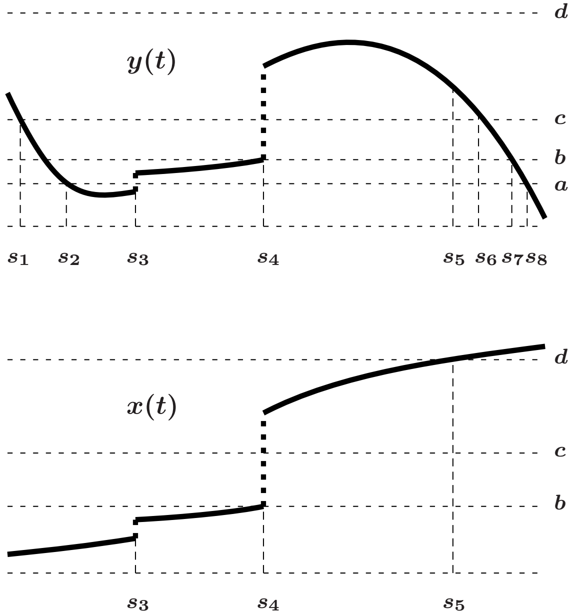


Figure 14: Relations between Convergences by Skorokhod (case $\mathbb{R}$)

M_2 : H-metric between $\mathcal{G}_F$ and $\mathcal{G}_G$

$M_1 + J_2 = J_1$



$$\nu_{[0,1]}^{[a,c]}[y(t)] = 3, \quad \nu_{[0,1]}^{[b,c]}[y(t)] = 3$$
$$\nu_{[0,1]}^{[a,d]}[y(t)] = 0$$
$$\nu_{[0,1]}^{[a,c]}[x(t)] = \nu_{[0,1]}^{[b,c]}[x(t)]$$
$$= \nu_{[0,1]}^{[a,d]}[x(t)] = 1$$

Figure 15: Examples of functions in $\mathcal{F}$ or $\mathcal{D}$

12. Computability on $\mathcal{F}$ and $\mathcal{F} \leftrightarrow \mathcal{P}$

(classical) convergence of $\{F_m\}$ to F

$F_m(x) \to F(x)$ for $\forall x$: point of continuity of F

Postulate I (Computability of pdf F) there is a computable sequence $\{x_n\}$ such that it is dense in $\mathbb{R}$

F is effectively continuous at x_n:

$$\exists \alpha(n,k) \text{ s.t. } |y - x_n| < 2^{-\alpha(n,k)} \Rightarrow |F(y) - F(x_n)| < 2^{-k}$$

$\{F(x_n)\}$ is computable

Postulate II (Effective convergene of $\{F_m\}$)

there is a computable sequence $\{x_n\}$ such that

it is dense in $\mathbb{R}$

each F_m is effectively continuous at x_n

$\{F_m(x_n)\}$ converges effectively uniformly at $\{x_n\}$

Proposition 12.1 *Suppose that there is a computable sequence $\{x_n\}$ such that it is dense in $\mathbb{R}$ and $\{F_m(x_n)\}$ is computable. Then $\{\mu_m\}$ is computable.*

Outline of the Proof. for a single F, μ and a single $f \in \mathcal{C}_\kappa$.

Take $\alpha(k)$ an effective modulus of uniform continuity of f and an integer L such that $f(x) = 0$ for $|x| \geq L$.

Let $y_{k,i} = -L + i2^{-\alpha(k+1)}$ $(0 \leq i \leq 2L2^{\alpha(k+1)} + 1)$. We can find effectively $n(k,i)$ such that $x_{n(k,i)} \in (y_{k,i-1}, y_{k,i})$.

Then $S_k = \sum_{i=1}^{2L2^{\alpha(k+1)}+1} f(x_{n(k,i)})(F(x_{n(k,i)}) - F(x_{n(k,i-1)}))$ converges effectively to $\int_{\mathbb{R}} f(x)\mu(dx)$.

$|\mu(f) - S_k| = \left|\sum_{i=1}^{2L2^{\alpha(k+1)}+1} \int_{(y_{k,i-1}, y_{k,i}]} (f(x) - f(x_{n(k,i)}))\mu(dx)\right| < 2^{-k}$. □

Proposition 12.2 *$\{\mu_m\}$ is computable and $\{x_n\}$ is computable. If F_m is continuous at $\{x_n\}$, then $F_m(x_n)$ is computable.*

Proof. $F(x - 2^{-n}) \leq \mu(\tilde{w}^-_{c,n}) \leq F(x) \leq \mu(\tilde{w}^+_{c,n}) \leq F(x - 2^{-n})$

$\{\mu(\tilde{w}^-_{c,n})\}$ and $\{\mu(\tilde{w}^+_{c,n})\}$ are computable

$\mu(\tilde{w}^-_{c,n}) - \mu(\tilde{w}^+_{c,n}) \downarrow 0 \Rightarrow$ convergence is effective □

Proposition 12.3 *Let $\mathcal{L}F$ be computable, x be computable and F be continuous at x. Then $F(x)$ is computable.*

Outline of the Proof:

On a neighborhood of x, $\tilde{F}$ is strictly increasing and $\hat{f}(t) = t - f(t)$ is its inverse.

$(f = \mathcal{L}F)$

$w^-_{x,n}$, $w^+_{x,n}$, and Proposition 4.4 may be useful.

$\mu(g) = \int_{\mathbb{R}} g(x)dF(x) = \int_{\mathbb{R}} g(\hat{f}(t))df(t)$ for all $g \in \mathcal{C}_\kappa$.

Lipschitz continuity implies $\leq \int_{\mathbb{R}} g(\hat{f})(t)dt$ if g is nonnegative.

Take $g = w^-_{x,n}, w^+_{x,n}$

$$\int_{\mathbb{R}} w^-_{x,n}(\hat{f}(t))df(t) = \mu(w^-_{x,n}) \le F(x) \le \mu(w^+_{x,n}) = \int_{\mathbb{R}} w^+_{x,n}(\hat{f}(t))df(t)$$

$$0 \le \int_{\mathbb{R}} w^+_{x,n}(\hat{f}(t))df(t) - \int_{\mathbb{R}} w^-_{x,n}(\hat{f}(t))df(t) = \int_{\mathbb{R}}\{w^+_{x,n}(\hat{f}(t)) - w^-_{x,n}(\hat{f}(t))\}df(t)$$

$$\le \int_{\mathbb{R}}\{w^+_{x,n}(\hat{f}(t)) - w^-_{x,n}(\hat{f}(t))\}dt = \int_{\tilde{F}(x-2^{-n})}^{\tilde{F}(x+2^{-n})}\{w^+_{x,n}(\hat{f}(t)) - w^-_{x,n}(\hat{f}(t))\}dt$$

$\rightarrow 0$ effectively (?) □

Conjecture 12.4 ***Suppose that there is a computable sequence $\{x_n\}$ such that it is dense in $\mathbb{R}$ and $\{F_m(x_n)\}$ is computable. Then $\{\mathcal{L}F_m\}$ is computable.***

If this conjecture is false, then we can say that Lévy computability is stronger than computability of the correspondint probability measures.

Other Facts

Fact 12.5 *Let $\{\mu_m\}$ and μ be computable, x be computable, F be continuous at x, and $\{\mu_m\}$ converge effectively to μ. Then $\{F_m(x)\}$ converges effectively uniformly to F at x.*
Or converges effectively continuously at x.

Fact 12.6 *Let $\{F_m\}$ and F be Lévy computable and x be computable. If $\{\mathcal{L}F_m\}$ converges effectively to $\mathcal{L}F$ then $\{F_m(x)\}$ converges effectively uniformly to F at x.*

Need to show $\int_{\mathbb{R}} g(\hat{f}(t))df(t)$ is computable if g is computable.
Put $g = u_{n,i}$.
effective convergence $\Leftrightarrow$ effective $\{g_n\}$-convergence $\Rightarrow$??

Fact 12.7 (classical) *$\{F_m\}$: a sequence of probability distribution functions, $\{F_m(x)\}$: converges (pointwise) to $F(x)$ at every continuity point of F
Then, $\{F_m(x)\}$ converges uniformly at every continuity point of F*

$(\because)$ $\forall\epsilon > 0\ \exists y_1 < x < y_2$ points of continuite of F, $|F(y_2) - F(y_1)| < \epsilon$
$F_m(y_1) \to F(y_1)$, $F_m(y_2) \to F(y_2)$,
$\exists N$ s.t. $m \geq N$, $|F_m(y_1) - F(y_1)| < \epsilon$, $|F_m(y_2) - F(y_2)| < \epsilon$
$m \geq N$ and $|y - x| < \min\{y_2 - x, x - y_1\}$, $|F_m(y) - F(x)| < 3\epsilon$

Other Conjectures

Conjecture 12.8 *Let $\{f_m\}$ is a computable sequence in $\mathcal{C}_L$ and converges monotonically to $f \in \mathcal{C}_L$. Then the convergence is effective and f is computable.*

Effective Dini Theorem 12.9 below by Kamo [7]

Theorem 12.9 Kamo [7]) *Let $(X, d, \mathcal{S})$ be an effectively compact metric space. Let $\{f_n\}$ be a computable sequence of real-valued functions on X and f a computable real-valued function on X. If $\{f_n\}$ converges pointwise monotonically to f as $n \to \infty$, then $\{f_n\}$ converges effectively uniformly to f.*

Conjecture 12.10 *If X is a effectively compact metric space, then $(\mathcal{P}(X), d_{v,\{g_n\}})$ is effectively compact for an effective separating set $\{g_n\}$ of $\mathcal{P}(X)$.*

Theorem 12.11 **(Effective decomposition of unity, Theorem 3 in [29])**
Let $\{O_n\}$ be an effective local finite r.e. covering of X. Then there exists a computable sequence of functions $\{f_n\}$ which satisfies
(i) $f_n(x) \geq 0$, (ii) $f_n(x) = 0$ *if* $x \notin O_n$, (iii) $\sum_{n=1}^{\infty} f_n(x) = 1$.

13. GADLAC

Definition 13.1 *A pair of functions $(\lambda(t), \xi(t))$ is said to be a parametric representation of the graph of $\mathcal{G}_F = \{(x, z) \mid F_-(x) \leq z \leq F(x),\ x \in \mathbb{R}\}$ if $\mathcal{G}_F = \{(\lambda(t), \xi(t) \mid t \in \mathbb{R}\}$, $\xi(t)$ is continuous and $\lambda(t)$ is continuous and monotonically increasing.*

Definition 13.2 (Skorokhod) $\{F_m\}$ *SM_1 converges to F: if there exist a parametric representation $(\lambda(t), \xi(t))$ of $\mathcal{G}_F$ and a sequence of parametric representations $(\lambda_m(t), \xi_m(t))$ of $\{\mathcal{G}_{F_m}\}$ respectively, such that*

$$\lim_{n\to\infty} \left\{\sup_t |\xi_m(t) - \xi(t)| + \sup_t |\lambda_m(t) - \lambda(t)|\right\} = 0$$

Postulate (1) A gadlac F is said to be SM_1- computable if there exist computable functions $(\lambda(t), \xi(t))$ which consist a parametric representation of $\mathcal{G}_F$. (computable parametric representation)
(2) $\{F_m\}$ SM_1-converges effectively to F: if there exist a computable parametric representation $(\lambda(t), \xi(t))$ of $\mathcal{G}_F$ and a computable sequence of parametric representations $(\lambda_m(t), \xi_m(t))$ of $\{\mathcal{G}_{F_m}\}$ respectively, such that $\lambda_m(t)$ and $\xi_m(t))$ converges effectively to $\lambda(t)$ and $\xi(t))$ respectively.

$\{F_n\}$ J_1-converges to F: There exists a sequence of continuous one-to-one and onto mappings $\{\lambda(x)\}$ such that

$$\lim_{n\to\infty} \sup_x |F_n(x) - F(\lambda_n(x))| = 0, \quad \lim_{n\to\infty} \sup_x |\lambda_n(x) - x| = 0. \tag{1}$$

This topology is the well known Skorokhod convergence. A equivalent metric which make complete the space of all gadlaces is discussed in Prokhorov [20] Appendix 1 and in Kolmogorov [9] (see also [1], [18]).
Since $F(\lambda(x))$ is not continuous, it seems difficult to define the corresponding notion of computability.

$\{F_m\}$ SJ_1 converges effectively to F: if there exist computable one-to-one and onto mapping $\{\lambda_m(t)\}$ such that $\sup_t |\lambda_m(t) - t|$ and $\sup_t |F_m - F(\lambda_m(t))|$ converge effectively to zero.

$\{F_n\}$ is said to converge uniformly to F at x: For all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\limsup_{n\to\infty} \sup_{|y-x|<\delta} |F_n(x) - F(x)| < \epsilon.$$

Well known properties

(1) If $\{F_n\}$ converges uniformly to F at any x in some closed finite interval $[a, b]$, then the convergence is uniform on $[a, b]$.

(2) $(\lambda_1(t), \xi_1(t))$ and $(\lambda_2(t), \xi_2(t))$ are parametric representations of some $\mathcal{G}_F$, then there exists a monotonically increasing function $u(t)$ such that $\lambda_1(t) = \lambda_2(u(t))$ and $\xi_1(t) = \xi(u(t))$.

? not continuous, generalized inverse

(3) If $\{F_n\}$ converges to F with respect to one of the four convergences, then $\{F_n\}$ converges uniformly o F at any point of continuity of F.

(4) Let Λ be the set of all continuous one-to-one and onto mappings from $[0, 1]$ to $[0, 1]$. Then,

$$\begin{aligned} d_S(F, G) &= \inf\Big\{\epsilon \,\Big|\, \exists \lambda \in \Lambda, \sup_t |t - \lambda(t)| < \epsilon, \sup_t |F(t) - G(\lambda(t))| < \epsilon\Big\} \\ &(=) \inf_{\lambda \in \Lambda}\Big\{\sup_t |F(t) - G(\lambda(t))| + \sup_t |t - \lambda(t)|\Big\} \end{aligned}$$

is called the Skorokhod metric. d_S-convergence and SJ_1-convergence are equivalent. $(\mathcal{D}([0, 1]), d_S)$ is not complete. (Skorokhod [24])

(5) Let $||\lambda|| = \sup_{s\neq t}\left|\log \frac{\lambda(t)-\lambda(s)}{t-s}\right|$ and

$$\tilde{d}_S(F,G) = \inf\left\{\epsilon \,\middle|\, \exists\lambda, ||\lambda|| < \epsilon, \sup_t |F(t) - G(\lambda(t))| < \epsilon\right\}.$$

Then, d_S and $\tilde{d}_S$ are equivalent. $(\mathcal{D}([0,1]), \tilde{d}_S)$ is complete.

(Prokhorov [20], Billingsley [1])

(6) Let $F_n(t) = \begin{cases} 0 & \text{if } t < \frac{1}{2} - \frac{1}{n} \\ \frac{n}{2}(t - \frac{1}{2}) + \frac{1}{2} & \text{if } \frac{1}{2} - \frac{1}{n} \leq t < \frac{1}{2} + \frac{1}{n} \\ 1 & \text{if } t \geq \frac{1}{2} + \frac{1}{n} \end{cases}.$

Then, $\{F_n\}$ SM_1-converges to $D_{\frac{1}{2}}$, but the convergence is not SJ_2. So, $\{F_n\}$ does not SJ_1-converges to $D_{\frac{1}{2}}$.

Conjecture 13.3 *Effective SM_1 convergence implies effective uniform convergence at any computable continuity point of F.*

14. Two dimensional probability measure

Necessary to handle with the Wasserstein Metric

Definition 14.1 The Lévy (Lévy-Prokhorov) metric $\tilde{d}_L(\mu,\nu) = \tilde{d}_L(F,G)$ is defined by

$$\tilde{d}_L(F,G) = \inf\{\epsilon > 0 \mid F(x-\epsilon, y-\epsilon) - \epsilon \le G(x,y) \le F(x+\epsilon, y+\epsilon) + \epsilon, \text{ for all } x\}$$

$\mathcal{G}_F = \{(x,y,z) \mid F(x_-, y_-) \le z \le F(x,y)\}$
$\ell_{s,t} = \{z + x = s\} \cap \{z + y = t\}$, $(s,t,v) = \mathcal{G}_F \cap \ell_{s,t}$
$\mathcal{L}F(s.t) := v$